Technical Supplement on Milgrom-Weber Auction Theory

Complementarity

Let $x \wedge y$, $x \vee y$, refer to the component-wise minima (*x meet y*) and maxima (*x join y*), respectively.

A function f:$R^n \rightarrow R$ is supermodular if $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$.

Remark: If f is twice-differentiable, then supermodularity reduces to:

$$i \neq j \text{ implies } \frac{\partial^2 f}{\partial x_i \, \partial x_j} \geq 0.$$

This, in turn, is equivalent to "increasing differences." That is, for $x_i > y_i$,

$f(x_1, \_, x_{i-1}, x_i, x_{i+1}, \_, x_n) - f(x_1, \_, x_{i-1}, y_i, x_{i+1}, \_, x_n)$ is nondecreasing in x_j for $j \neq i$.

If f is a payoff function, the variables of f are said to be complementary.

Affiliation

If the function log f is supermodular, f is said to be log supermodular [log-spm]. If f is a density, then the random variables with density f are said to be *affiliated*. If there are two of them, f is said to have the *monotone likelihood ratio property* (MLRP).

(i) Affiliation is equivalent to the statement that $E[\alpha(\mathbf{X})|a_i \leq X_i \leq b_i]$ is nondecreasing in a_i, b_i for all nondecreasing functions α.

Proof: Consider $\_(y)$=E[$\alpha(x,Y)|Y=y, a \leq x \leq b$]. Below, expectations refer to conditioning on $Y=y$, $a \leq x \leq b$.

$$j'(y) = E[\frac{\partial a}{\partial y}] + \int a(x,y) \left[\frac{f_y(x|y)}{\int_a^b f(z|y)\,dz} - \frac{f(x|y) \int_a^b f_y(z|y)\,dz}{\left(\int_a^b f(z|y)\,dz \right)^2} \right] dx$$

$$= E[\frac{\partial a}{\partial y}] + E[a(x,y)\frac{f_y(x|y)}{f(x|y)}] - E[a(x,y)]E[\frac{f_y}{f}]$$

$$= E[\frac{\partial a}{\partial y}] + cov(a, \frac{f_y}{f}).1$$

If the MLRP is satisfied, this is nonnegative. Conversely, let α be increasing in x and constant in y. Then _ is nondecreasing for all y, a, and b if and only if the MLRP is satisfied.

(ii) Nondecreasing functions of affiliated r.v.'s are affiliated (see Milgrom-Weber).

Let x, y have density $f(x,y)$, and denote the density of y given x by $f_Y(y|x)$, with cdf $F_Y(y|x)$.

(iii) $F_Y(y|x)$ is nonincreasing in x (First Order Stochastic Dominance).

The characteristic function of a set, 1_A, is the function which is 1 if $x \in A$ and 0 otherwise. Note $Pr[X_i \geq x_i] = E[1_{\{X_i \geq x_i\}}]$. It follows that $\Pr[X_i \geq x_i \mid X_j = x_j]$ is nondecreasing in x_j.

(iv) f is log-spm if and only if f_Y is log-spm.

Proof: $\frac{\partial^2}{\partial x\ \partial y}\log\ f_Y(\ y \mid x) = \frac{\partial^2}{\partial x\ \partial y}\log\left(\frac{f(x,y)}{\int f(x,z)\,dz}\right)$

$$\frac{\partial^2}{\partial x\ \partial y}\log\ f(x,y) - \frac{\partial^2}{\partial x\ \partial y}\log\left(\int f(x,z)\,dz\right) = \frac{\partial^2}{\partial x\ \partial y}\log\ f(x,y).$$

(v) Independently distributed random variables are affiliated.

(vi) If $f(y|x)$ is log-spm, $F(y|x)$ is log-spm

Proof: $\frac{\partial}{\partial x}\frac{f(y \mid x)}{F(y \mid x)} = \frac{f_2(y \mid x)}{F(y \mid x)} - \frac{f(y \mid x)\ F_2(y \mid x)}{F(y \mid x)^2}$

$$= \frac{f(y \mid x)}{F(y \mid x)^2}\left[\frac{f_2(y \mid x)}{f(y \mid x)}F(y \mid x) - F_2(y \mid x)\right]$$

$$= \frac{f(y \mid x)}{F(y \mid x)^2}\left[\int_0^y\left(\frac{f_2(y \mid x)}{f(y \mid x)} - \frac{f_2(z \mid x)}{f(z \mid x)}\right)f(z \mid x)\ dz\right] \geq 0.$$

(vii) if f,g are log-spm, fg is log-spm. Proof is $\log fg = \log f + \log g$.

Auction Environment

Bidder i privately receives a signal that is the realization of the r.v. X_i; the vector $(X_1,\ldots,X_n,\boldsymbol{S})$ are affiliated and the X_i's are symmetrically distributed. The payoff to bidder i is $u(X_i,X_{-i},\mathbf{S})$. u is assumed nondecreasing in all arguments. We fix attention on bidder 1 and let $Y = \max\{X_2,\ldots,X_n)$. Y is affiliated with X_1. Let $f_Y(Y|x)$ be the density of Y given $X_1=x$, with distribution function F_Y. Let $v(x,y)=E[u|x_1=x,Y=y]$. Since u is nondecreasing, so is v.

Second Price Auction

In a second price auction, the high bidder obtains the object and pays the second highest bid.

A symmetric equilibrium bidding function is a function B_2 such that, given all other bidders bid according to B_2, the remaining bidder maximizes expected profit by bidding $B_2(x)$ given signal x. Consider bidder 1 with signal x who instead bids $B_2(z)$. This bidder earns

$$p = \int_0^z (v(x,y) - B_2(y)) \; f_Y(y \mid x) \; dy.$$

In order for B_2 to be an equilibrium, π must be maximized at z=x, which implies

$$B_2(x) = v(x,x).$$

It is straightforward to show that B_2 is indeed an equilibrium, and is the only symmetric equilibrium.

If a reserve price (minimum acceptable bid) r is imposed, bidders with signals below x_r, where $E[v(x_r,Y)|Y \leq x_r]=r$, do not submit bids; otherwise the equilibrium is unperturbed. Note however, that the minimum submitted bid, $B_2(x_r) > r$!

Suppose the seller knows S_i. Should the seller tell the bidders $\boldsymbol{S}$? Let

$$w(x,y,s) = E[u \mid X_1=x, Y=y, S_i=s].$$

$$v(y,y) = E[w(X_1,Y,S_i) \mid X_1 = Y = y]$$

$$= E[w(Y,Y,S_i) \mid X_1 = Y = y]$$

$$\leq E[w(Y,Y,S_i) \mid X_1 \geq Y = y].$$

The seller's revenue with no disclosure, R_N, is

$$R_N = E[v(Y,Y) \mid X_1 \geq Y]$$

$$\leq E[E[w(Y,Y,S_i) \mid X_1 \geq Y] \mid x_1 > Y]$$

$$= E[w(Y,Y,S_i) \mid X_1 > Y] = R_I \text{, the revenue with disclosure of } S_i.$$

First Price Auction

In a first price auction, the high bidder obtains the object and pays her bid.

Suppose B_1 is a symmetric equilibrium. The profits to bidder 1, with signal *x*, who bids $B_1(z)$, are:

$$p = \int_0^z (v(x,y) - B(z)) \; f_Y(y \mid x) \; dy.$$

Maximizing with respect to *z*, and setting *z*=*x*, yields the first order differential equation

$$B_1'(x) = \frac{f_Y(x \mid x)}{F_Y(x \mid x)}(v(x,x) - B_1(x)).$$

Suppose that the reserve price is zero. Then the differential equation has solution

$$B_1(x) = \int_0^x e^{-\int_y^x \frac{f_Y(z \mid z)}{F_Y(z \mid z)} dz} \frac{f_Y(y \mid y)}{F_Y(y \mid y)} \; v(y,y) \; dy.$$

If the reserve price $r > 0$, the screening level is x_r and B_1 satisfies $B_1(x_r)=r$.

Integrating $B_1(x)$ by parts, we have:

$$B_1(x) = v(x,x) - \int_0^x e^{-\int_y^x \frac{f_Y(z \mid z)}{F_Y(z \mid z)} dz} \left[\frac{d}{dy} v(y,y) \right] dy.$$

Conditional on winning with a signal of *x* (probability $F_Y(x|x)$), a bidder in a second price auction pays

$$EB_2 = \int_0^x v(y,y) \frac{f_Y(y|x)}{F_Y(x \mid x)} \; dy = v(x,x) - \int_0^x \frac{F_Y(y \mid x)}{F_Y(x \mid x)} \left[\frac{d}{dy} v(y,y) \right] dy.$$

Note that $\log F_Y(x \mid x) - \log F_Y(y \mid x) = \int_y^x \frac{f_Y(z|x)}{F_Y(z|x)} dz \geq \int_y^x \frac{f_Y(z|z)}{F_Y(z|z)} dz,$ (by(vi)) and thus, $\frac{F_Y(y \mid x)}{F_Y(x \mid x)} \leq e^{-\int_y^x \frac{f_Y(z \mid z)}{F_Y(z \mid z)} dz}$.

Since *v* is nondecreasing, the expected payment by a winning bidder with signal *x* is higher in a second price auction than in a first price auction.